

PERIODICITY OF FREE SUBGROUP NUMBERS MODULO PRIME POWERS

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ABSTRACT. We completely characterise when the sequence of free subgroup numbers of a finitely generated virtually free group is ultimately periodic modulo a given prime power.

1. INTRODUCTION

For a finitely generated virtually free group Γ , denote by m_Γ the least common multiple of the orders of the finite subgroups in Γ and, for a positive integer λ , let $f_\lambda(\Gamma)$ denote the number of free subgroups of index λm_Γ in Γ . In [9], the authors show, among other things, that the number $f_\lambda(PSL_2(\mathbb{Z}))$ of free subgroups of index 6λ in the inhomogeneous modular group $PSL_2(\mathbb{Z})$, considered as a sequence indexed by λ , is ultimately periodic modulo any fixed prime power p^α , if p is a prime number with $p \geq 5$. More precise results on the length of the period, and an explicit formula for the linear recurrence satisfied by these numbers modulo p^α are also provided in [9]. As is well known, ultimate periodicity of the sequence $(f_\lambda(\Gamma))_{\lambda \geq 1}$ is equivalent to rationality of the corresponding generating function $F_\Gamma(z) = \sum_{\lambda \geq 0} f_{\lambda+1}(\Gamma) z^\lambda$.

The purpose of the present paper is to demonstrate that the periodicity phenomenon discovered in [9] holds in a much wider context, namely that of finitely generated virtually free groups. Indeed, our main result (Theorem 1) provides an explicit characterisation of all pairs (Γ, p^α) , where Γ is a finitely generated virtually free group and p^α is a proper prime power, for which the sequence of free subgroup numbers of Γ is ultimately periodic modulo p^α . Roughly speaking, for “almost all” pairs (Γ, p) the sequence $(f_\lambda(\Gamma))_{\lambda \geq 0}$ is ultimately periodic modulo p^α for all $\alpha \geq 1$, the only exception occurring when $p \mid m_\Gamma$ and $\mu_p(\Gamma) = 0$, where $\mu_p(\Gamma)$ is a certain invariant defined in (2.9) and discussed in the paragraph following that formula.

In order to further place our results into context, we point out that, for primes p dividing the constant m_Γ , an elaborate theory is presented in [16] for the behaviour of the arithmetic function $f_\lambda(\Gamma)$ modulo p . Recently, this theory has been supplemented by congruences modulo (essentially arbitrary) 2-powers and 3-powers for the number of free subgroups of finite index in lifts of the classical modular group; that is, amalgamated products of the form

$$\Gamma_\ell = C_{2\ell} \underset{C_\ell}{*} C_{3\ell}, \quad \ell \geq 1;$$

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cf. Theorems 19 and 20 in [8, Sec. 8], and Section 16 in [10], in particular, [10, Thms. 49–52]. These results demonstrate a highly non-trivial behaviour of the sequences $(f_\lambda(\Gamma_\ell))_{\lambda \geq 1}$ modulo powers of 2 if ℓ is odd (in which case $\mu_2(\Gamma_\ell) = 0$), and modulo powers of 3 for $3 \nmid \ell$ (in which case $\mu_3(\Gamma_\ell) = 0$). For instance, for the sequence $(f_\lambda = f_\lambda(\Gamma_1))_{\lambda \geq 1}$ of free subgroup numbers of the group $\mathrm{PSL}_2(\mathbb{Z})$, one obtains that:

- (1) $f_\lambda \equiv -1 \pmod{3}$ if, and only if, the 3-adic expansion of λ is an element of $\{0, 2\}^*1$;
- (2) $f_\lambda \equiv 1 \pmod{3}$ if, and only if, the 3-adic expansion of λ is an element of

$$\{0, 2\}^*100^* \cup \{0, 2\}^*122^*;$$

- (3) for all other λ , we have $f_\lambda \equiv 0 \pmod{3}$;

cf. [10, Cor. 53]. Here, for a set Ω , we denote by Ω^* the free monoid generated by Ω . All this is in sharp contrast to “most” of the cases in the classification result in Theorem 1, which exhibit “simple” (ultimate) periodicity.

In proving Theorem 1, the bulk of the argument lies in showing that, if p is a prime number *not* dividing m_Γ , then the sequence $(f_\lambda(\Gamma))_{\lambda \geq 1}$ is ultimately periodic modulo p^α for every integer $\alpha \geq 1$; this is the contents of Theorem 2, whose proof occupies Sections 4–10. The case where $p \mid m_\Gamma$ is largely taken care of by Theorem 3; its proof in Section 11 is by an inductive argument, which is based on an earlier generating function result in [16]. The proof of Theorem 1 itself appears in Section 12. Precise formulations of our results are found in Section 3, while the next section collects together definitions as well as some background material on virtually free groups.

2. SOME PRELIMINARIES ON VIRTUALLY FREE GROUPS

Our notation and terminology here follows Serre’s book [22]; in particular, the category of graphs used is described in [22, §2]. This category deviates slightly from the usual notions in graph theory. Specifically, a *graph* X consists of two sets: $E(X)$, the set of (directed) *edges*, and $V(X)$, the set of *vertices*. The set $E(X)$ is endowed with a fixed-point-free involution $\bar{} : E(X) \rightarrow E(X)$ (*reversal of orientation*), and there are two functions $o, t : E(X) \rightarrow V(X)$ assigning to an edge $e \in E(X)$ its *origin* $o(e)$ and *terminus* $t(e)$, such that $t(\bar{e}) = o(e)$. The reader should note that, according to the above definition, graphs may have loops (that is, edges e with $o(e) = t(e)$) and multiple edges (that is, several edges with the same origin and the same terminus). An *orientation* $\mathcal{O}(X)$ consists of a choice of exactly one edge in each pair $\{e, \bar{e}\}$ (this is indeed always a *pair* – even for loops – since, by definition, the involution $\bar{}$ is fixed-point-free). Such a pair is called a *geometric edge*.

Let Γ be a finitely generated virtually free group with Stallings decomposition $(\Gamma(-), X)$; that is, $(\Gamma(-), X)$ is a finite graph of finite groups with fundamental group $\Gamma \cong \pi_1(\Gamma(-), X)$. If \mathfrak{F} is a free subgroup of finite index in Γ then, following an idea of C. T. C. Wall, one defines the (rational) Euler characteristic $\chi(\Gamma)$ of Γ as

$$\chi(\Gamma) = -\frac{\mathrm{rk}(\mathfrak{F}) - 1}{(\Gamma : \mathfrak{F})}. \quad (2.1)$$

(This is well-defined in view of Schreier's index formula in [19].) In terms of the above decomposition of Γ , we have

$$\chi(\Gamma) = \sum_{v \in V(X)} \frac{1}{|\Gamma(v)|} - \sum_{e \in \mathcal{O}(X)} \frac{1}{|\Gamma(e)|}. \quad (2.2)$$

Equation (2.2) reflects the fact that, in our situation, the Euler characteristic in the sense of Wall coincides with the equivariant Euler characteristic $\chi_T(\Gamma)$ of Γ relative to the tree T canonically associated with Γ in the sense of Bass–Serre theory; cf. [1, Chap. IX, Prop. 7.3] or [21, Prop. 14]. We remark that a finitely generated virtually free group Γ is largest among finitely generated groups in the sense of Pride's preorder [18] (i.e., Γ has a subgroup of finite index, which can be mapped onto the free group of rank 2) if, and only if, $\chi(\Gamma) < 0$; see Lemma 12 in Section 8.

As in the introduction, denote by m_Γ the least common multiple of the orders of the finite subgroups in Γ , so that, again in terms of the above Stallings decomposition of Γ ,

$$m_\Gamma = \text{lcm}\{|\Gamma(v)| : v \in V(X)\}.$$

(This formula essentially follows from the well-known fact that a finite group has a fixed point when acting on a tree.) The type $\tau(\Gamma)$ of a finitely generated virtually free group $\Gamma \cong \pi_1(\Gamma(-), X)$ is defined as the tuple

$$\tau(\Gamma) = (m_\Gamma; \zeta_1(\Gamma), \dots, \zeta_\kappa(\Gamma), \dots, \zeta_{m_\Gamma}(\Gamma)),$$

where the $\zeta_\kappa(\Gamma)$'s are integers indexed by the divisors of m_Γ , given by

$$\zeta_\kappa(\Gamma) = |\{e \in \mathcal{O}(X) : |\Gamma(e)| \mid \kappa\}| - |\{v \in V(X) : |\Gamma(v)| \mid \kappa\}|.$$

It can be shown that the type $\tau(\Gamma)$ is in fact an invariant of the group Γ , i.e., independent of the particular decomposition of Γ in terms of a graph of groups $(\Gamma(-), X)$, and that two finitely generated virtually free groups Γ_1 and Γ_2 contain the same number of free subgroups of index n for each positive integer n if, and only if, $\tau(\Gamma_1) = \tau(\Gamma_2)$; cf. [14, Theorem 2]. We have $\zeta_\kappa(\Gamma) \geq 0$ for $\kappa < m_\Gamma$ and $\zeta_{m_\Gamma}(\Gamma) \geq -1$ with equality occurring in the latter inequality if, and only if, Γ is the fundamental group of a tree of groups; cf. [13, Prop. 1] or [14, Lemma 2].

We observe that, as a consequence of (2.2), the Euler characteristic of Γ can be expressed in terms of the type $\tau(\Gamma)$ via

$$\chi(\Gamma) = -m_\Gamma^{-1} \sum_{\kappa \mid m_\Gamma} \varphi(m_\Gamma/\kappa) \zeta_\kappa(\Gamma), \quad (2.3)$$

where φ is Euler's totient function. It follows in particular that, if two finitely generated virtually free groups have the same number of free subgroups of index n for every n , then their Euler characteristics must coincide.

The proof of Theorem 2, as given in Section 10, is based on the analysis of a second arithmetic function associated with the group Γ . Define a *torsion-free Γ -action on a set Ω* to be a Γ -action on Ω which is free when restricted to finite subgroups, and let

$$g_\lambda(\Gamma) := \frac{\text{number of torsion-free } \Gamma\text{-actions on a set with } \lambda m_\Gamma \text{ elements}}{(\lambda m_\Gamma)!}, \quad \lambda \geq 0;$$

in particular, $g_0(\Gamma) = 1$. The sequences $(f_\lambda(\Gamma))_{\lambda \geq 1}$ and $(g_\lambda(\Gamma))_{\lambda \geq 0}$ are related via the Hall-type transformation formula¹

$$\sum_{\mu=0}^{\lambda-1} g_\mu(\Gamma) f_{\lambda-\mu}(\Gamma) = m_\Gamma \lambda g_\lambda(\Gamma), \quad \lambda \geq 1. \quad (2.4)$$

Moreover, a careful analysis of the universal mapping property associated with the presentation $\Gamma \cong \pi_1(\Gamma(-), X)$ leads to the explicit formula

$$g_\lambda(\Gamma) = \frac{\prod_{e \in \mathcal{O}(X)} (\lambda m_\Gamma / |\Gamma(e)|)! |\Gamma(e)|^{\lambda m_\Gamma / |\Gamma(e)|}}{\prod_{v \in V(X)} (\lambda m_\Gamma / |\Gamma(v)|)! |\Gamma(v)|^{\lambda m_\Gamma / |\Gamma(v)|}}, \quad \lambda \geq 0, \quad (2.5)$$

for $g_\lambda(\Gamma)$, where $\mathcal{O}(X)$ is any orientation of X ; cf. [14, Prop. 3]. Introducing the generating functions

$$F_\Gamma(z) := \sum_{\lambda \geq 0} f_{\lambda+1}(\Gamma) z^\lambda \quad \text{and} \quad G_\Gamma(z) := \sum_{\lambda \geq 0} g_\lambda(\Gamma) z^\lambda,$$

Equation (2.4) is seen to be equivalent to the relation

$$F_\Gamma(z) = m_\Gamma \frac{d}{dz} (\log G_\Gamma(z)). \quad (2.6)$$

Define the *free rank* $\mu(\Gamma)$ of a finitely generated virtually free group Γ to be the rank of a free subgroup of index m_Γ in Γ (existence of such a subgroup follows, for instance, from Lemmas 8 and 10 in [22]; it need not be unique, though). We note that, in view of (2.1), the quantity $\mu(\Gamma)$ is connected with the Euler characteristic of Γ via

$$\mu(\Gamma) + m_\Gamma \chi(\Gamma) = 1, \quad (2.7)$$

which shows in particular that $\mu(\Gamma)$ is well-defined. Combining Equations (2.3) and (2.7), we see that the free rank $\mu(\Gamma)$ can be expressed in terms of the type of Γ via

$$\mu(\Gamma) = 1 + \sum_{\kappa | m_\Gamma} \varphi(m_\Gamma / \kappa) \zeta_\kappa(\Gamma). \quad (2.8)$$

Given a finitely generated virtually free group Γ and a prime number p , we introduce, in analogy with formula (2.8), the p -rank $\mu_p(\Gamma)$ of Γ via the equation

$$\mu_p(\Gamma) = 1 + \sum_{p | \kappa | m_\Gamma} \varphi(m_\Gamma / \kappa) \zeta_\kappa(\Gamma). \quad (2.9)$$

Clearly, $\mu_p(\Gamma) \geq 0$, with equality occurring in this inequality if, and only if, Γ is the fundamental group of a tree of groups and $\zeta_\kappa(\Gamma) = 0$ for $p \mid \kappa \mid m_\Gamma$ and $\kappa < m_\Gamma$. Similarly, we have $\mu_p(\Gamma) = 1$ if, and only if, (i) $\zeta_\kappa(\Gamma) = 0$ for all κ with $p \mid \kappa \mid m_\Gamma$, or (ii) Γ is the fundamental group of a tree of groups, m_Γ is even, $p \mid m_\Gamma/2$, $\zeta_{m_\Gamma/2}(\Gamma) = 1$, and $\zeta_\kappa(\Gamma) = 0$ for $p \mid \kappa \mid m_\Gamma$ and $\kappa < m_\Gamma/2$. To give concrete examples, if p is an odd prime number, then the groups

$$\Gamma_{p,\alpha} = C_2 * C_{2p} * \underbrace{C_p * \cdots * C_p}_{\alpha \text{ copies}}, \quad \alpha \geq 0$$

¹See [14, Cor. 1], or [4, Prop. 1] for a more general result.

satisfy $\mu_p(\Gamma_{p,\alpha}) = 1$, while the groups

$$\Gamma_{2,\alpha} = C_4 * C_4 * \underbrace{C_2 * \cdots * C_2}_{\alpha \text{ copies}}, \quad \alpha \geq 0$$

satisfy $\mu_2(\Gamma) = 1$.

3. THE RESULTS

Here and in the sequel, given power series $f(z)$ and $g(z)$, we write

$$f(z) = g(z) \text{ modulo } p^\gamma$$

to mean that the coefficients of z^i in $f(z)$ and $g(z)$ agree modulo p^γ for all i ; in particular, the phrase “ $F_\Gamma(z)$ is rational modulo p^α ” means that $F_\Gamma(z)$ equals a certain rational function modulo p^α in the sense of the above definition. Our main result, which completely characterises rationality of the generating function $F_\Gamma(z)$ (i.e., ultimate periodicity of the sequence $(f_\lambda(\Gamma))_{\lambda \geq 1}$) modulo prime powers, is as follows.

Theorem 1. *Let Γ be a finitely generated virtually free group, let p be a prime number, and let $F_\Gamma(z)$ denote the generating function $\sum_{\lambda \geq 0} f_{\lambda+1}(\Gamma)z^\lambda$ for the free subgroup numbers of Γ . Then the following assertions are equivalent:*

- (i) *the series $F_\Gamma(z)$ is rational modulo p^α for each positive integer α ;*
- (ii) *the series $F_\Gamma(z)$ is rational modulo p ;*
- (iii) *The pair (Γ, p) satisfies one of the following mutually exclusive conditions:*
 - (iii)₁ *$p \nmid m_\Gamma$;*
 - (iii)₂ *$p \mid m_\Gamma$ and $\mu_p(\Gamma) > 0$;*
 - (iii)₃ *Γ is finite;*
 - (iii)₄ *Γ is virtually infinite-cyclic and $p = 2$.*

The proof of Theorem 1 is broken up into two main steps, each of which is a meaningful result in its own right. Case (iii)₁ is taken care of by the following fundamental result.

Theorem 2. *Let Γ be a finitely generated virtually free group, let p be a prime number not dividing m_Γ , and let α be a positive integer. Then the sequence $(f_\lambda(\Gamma))_{\lambda \geq 1}$ is ultimately periodic modulo p^α .*

The case (iii)₂, where $p \mid m_\Gamma$ and $\mu_p(\Gamma) > 0$, is dealt with in our last result.

Theorem 3. *Let Γ be a finitely generated virtually free group, let p be a prime number such that $p \mid m_\Gamma$ and $\mu_p(\Gamma) > 0$, and let α be a positive integer. Then the generating function $F_\Gamma(z)$ for the free subgroup numbers of Γ is rational modulo p^α . If $\mu_p(\Gamma) = 1$, then $F_\Gamma(z)$ is non-polynomial modulo p^α and the rational fraction can be written so that the denominator is a power of $1-z$ or $1+z$, depending on whether $(\mu(\Gamma) - \mu_p(\Gamma))/(p-1)$ is even or odd. If $\mu_p(\Gamma) \geq 2$, then $F_\Gamma(z)$ is polynomial modulo p^α .*

Other ingredients in the proof of Theorem 1 are Corollary 10, which describes the function $f_\lambda(\Gamma)$ in the case where Γ is virtually infinite-cyclic, as well as [16, Prop. 2] and [16, Theorem 2].

The proof of Theorem 2, as given in Section 10, is based on the analysis of a second, rational-valued, arithmetic function associated with the group Γ , whose λ -th term may be viewed as the “normalised” number of torsion-free Γ -actions on a set with λm_Γ elements. This function, denoted here by $g_\lambda(\Gamma)$, is connected with the free subgroup numbers $f_\lambda(\Gamma)$ via the crucial equation (2.4); see Section 2. A careful p -adic analysis of this equation will show in the end that the numbers $f_\lambda(\Gamma)$ satisfy a linear recurrence modulo any fixed prime power p^α with p not dividing m_Γ . By standard results on linear recurring sequences, Theorem 2 then follows immediately.

The function $g_\lambda(\Gamma)$ has been discussed in Section 2, together with some further preliminaries on virtually-free groups. Sections 4–9 prepare for the proof of Theorem 2 in Section 10: Section 4 surveys the relevant results from the theory of linear recurring sequences; Section 5 provides a representation of a finitely generated virtually free group in terms of a finite graph of finite groups without trivial amalgamations, a representation which is particularly suited for our subsequent analysis; Section 6 contains a purely graph-theoretic lemma on orientation of trees; Section 7 discusses the classification of virtually free groups of (free) rank at most 2; Section 8 collects together criteria for finitely generated virtually free groups to be ‘large’, while Section 9 establishes a crucial p -divisibility property for this second arithmetic function $g_\lambda(\Gamma)$.

We conclude this section with some remarks.

Remarks 4. (1) It is shown in [16] that, if Γ is a finitely generated virtually free group with $\mu_p(\Gamma) = 0$ for a given prime p , then the function $f_\lambda(\Gamma)$ satisfies the congruence

$$f_\lambda(\Gamma) \equiv (-1)^{\frac{(\mu(\Gamma)-1)}{p-1}} \lambda^{-1} \left(\frac{\frac{\mu(\Gamma)\lambda}{p-1}}{\frac{\lambda-1}{p-1}} \right) \pmod{p};$$

cf. [16, Eqn. (35)]. In general, it remains an open problem, how the free subgroup numbers of a finitely generated virtually free group Γ with $\mu_p(\Gamma) = 0$ behave modulo higher p -powers. The only results known in this direction concern (i) lifts of Hecke groups $\mathfrak{H}(q) \cong C_2 * C_q$ with q a Fermat prime and $p = 2$, and (ii) lifts of the classical modular group $\mathfrak{H}(3) \cong PSL_2(\mathbb{Z})$ with $p = 3$; see Corollary 34 and Theorem 35 in [8], and [10, Sec. 16].

(2) By a *cyclic cover*, we mean the fundamental group Γ of a finite graph $(\Gamma(-), X)$ of finite cyclic groups. To fix ideas, we shall assume that the canonical embeddings associated with $(\Gamma(-), X)$ are induced by the identity maps of the corresponding vertex stabilisers. Let $\Gamma = \pi_1(\Gamma(-), X)$ be a cyclic cover, and let ℓ be a positive integer. Then we define the ℓ -th lift Γ_ℓ of Γ as the cyclic cover resulting from $(\Gamma(-), X)$ by multiplying the order of each (vertex or edge) stabiliser by a factor ℓ . The last assertion in Theorem 3 implies that $F_{\Gamma_\ell}(z)$ is polynomial modulo all proper p -powers for all lifts Γ_ℓ of cyclic covers Γ with $p \nmid m_\Gamma$, $\mu(\Gamma) \geq 2$, and $p \mid \ell$.

(3) In order to illustrate Theorem 3, let us consider the case where $\Gamma = \mathfrak{H}(6) \cong C_2 * C_6$ and $p = 3$. Indeed, in this example, we have $3 \mid m_{\mathfrak{H}(6)} = 6$ and $\mu_3(\mathfrak{H}(6)) = 1$. If one applies the algorithm which is implicit in the proof of Theorem 3 given in Section 11,

then, modulo $3^9 = 19683$, one obtains

$$F_{\mathfrak{H}(6)}(z) = \frac{1}{(1+z)^{10}}(19680z^9 + 585z^8 + 1926z^7 + 6165z^6 + 7326z^5 \\ + 1584z^4 + 1566z^3 + 17433z^2 + 1845z + 15) \quad \text{modulo } 3^9.$$

Consequently, the period length of the ultimately periodic sequence $(f_\lambda(\mathfrak{H}(6)))_{\lambda \geq 1}$ is $2 \cdot 3^{11} = 354294$ when taken modulo 3^9 .

(4) For a finitely generated virtually free group Γ , denote by $\hat{\Gamma}$ the isomorphism class of Γ . Given a prime number p , define a density \mathfrak{D}_p of isomorphism classes of groups Γ with $\mu_p(\Gamma) = 0$ in all isomorphism classes by

$$\mathfrak{D}_p := \lim_{M \rightarrow \infty} \frac{|\{\hat{\Gamma} : m_\Gamma \leq M, \mu(\Gamma) \leq M, \mu_p(\Gamma) = 0\}|}{|\{\hat{\Gamma} : m_\Gamma \leq M, \mu(\Gamma) \leq N\}|}.$$

Note that this definition makes sense in view of [14, Prop. 4] and Equation (2.7). We conjecture that $\mathfrak{D}_p = 0$ for all prime numbers p .

4. PERIODICITY OF SEQUENCES OVER FINITE RINGS

In this section we review some standard results on linear recurring sequences (usually only formulated over finite fields), which will be used in a crucial manner in the proof of Theorem 2 in Section 10.

Let Λ be a finite commutative ring with identity, and let $\mathcal{S} = (s_n)_{n \geq 0}$ be a sequence of elements of Λ . Suppose that there exist a positive integer d and elements $\alpha_0, \alpha_1, \dots, \alpha_{d-1}, \alpha \in \Lambda$, such that \mathcal{S} satisfies the relation

$$s_{n+d} = \alpha_{d-1}s_{n+d-1} + \alpha_{d-2}s_{n+d-2} + \dots + \alpha_0s_n + \alpha, \quad n \geq 0. \quad (4.1)$$

Then \mathcal{S} is termed a *linear recurring sequence* over Λ of order d , a relation of the form (4.1) itself is called a *linear recurrence relation* (or difference relation) of order d . Relation (4.1) is called *homogeneous* if $\alpha = 0$, otherwise *inhomogeneous*; the sequence \mathcal{S} itself is called a *homogeneous*, or *inhomogeneous*, *linear recurring sequence* over Λ , respectively.

The sequence $\mathcal{S} = (s_n)_{n \geq 0}$ is termed *ultimately periodic*, if there exist integers $\omega > 0$ and $n_0 \geq 0$, such that $s_{n+\omega} = s_n$ holds for all $n \geq n_0$. The integer ω is then called a *period* of \mathcal{S} . The smallest number among all the possible periods ω of an ultimately periodic sequence \mathcal{S} is called the *least period* $\omega_0 = \omega_0(\mathcal{S})$ of \mathcal{S} . If $\mathcal{S} = (s_n)_{n \geq 0}$ is ultimately periodic with least period ω_0 , then the least non-negative integer n_0 such that $s_{n+\omega_0} = s_n$ for all $n \geq n_0$ is called the *preperiod* of \mathcal{S} . An ultimately periodic sequence $\mathcal{S} = (s_n)_{n \geq 0}$ with least period $\omega_0(\mathcal{S})$ is termed *purely periodic*, if $s_{n+\omega_0(\mathcal{S})} = s_n$ for all $n \geq 0$. It is easy to see that a sequence $\mathcal{S} = (s_n)_{n \geq 0}$ is purely periodic, if, and only if, there exists an integer $\omega > 0$ such that $s_{n+\omega} = s_n$ for all $n \geq 0$. Also, every period of an ultimately periodic sequence is divisible by the least period.

Linear recurring sequences over finite rings are always (ultimately) periodic. The following result, which concentrates on the case of a homogeneous linear recurring sequences, will suffice for our present purposes.

Lemma 5. *Let $\mathcal{S} = (s_n)_{n \geq 0}$ be a homogeneous linear recurring sequence of order $d \geq 1$ over a finite commutative ring Λ with identity. Then \mathcal{S} is ultimately periodic with least period $\omega_0(\mathcal{S}) < |\Lambda|^d$. Moreover, if the linear recurrence relation (4.1) satisfied by \mathcal{S} is such that α_0 is invertible (i.e., a unit of Λ), then \mathcal{S} is purely periodic.*

The proof of Lemma 5 is virtually identical with that in the case of finite fields; see Chapter 8 in [11], in particular Theorems 8.7 and 8.11, and Lemma 8.12.

5. NORMALISING A FINITE GRAPH OF GROUPS

It will be helpful to be able to represent a finitely generated virtually free group Γ by a graph of groups avoiding trivial amalgamations. This is achieved via the following.

Lemma 6 (NORMALISATION). *Let $(\Gamma(-), X)$ be a (connected) graph of groups with fundamental group Γ , and suppose that X has only finitely many vertices. Then there exists a graph of groups $(\Delta(-), Y)$ with $|V(Y)| < \infty$ and a spanning tree T in Y , such that $\pi_1(\Delta(-), Y) \cong \Gamma$, and such that²*

$$\Delta(e)^e \neq \Delta(t(e)) \text{ and } \Delta(e)^{\bar{e}} \neq \Delta(o(e)), \quad \text{for } e \in E(T). \quad (5.1)$$

Moreover, if $(\Gamma(-), X)$ satisfies the finiteness condition

$$(F_1) \quad X \text{ is a finite graph,}$$

or

$$(F_2) \quad \Gamma(v) \text{ is finite for every vertex } v \in V(X),$$

then we may choose $(\Delta(-), Y)$ so as to enjoy the same property.

Proof. Choose a spanning tree S in X , and call an edge $e \in E(S)$ *trivial*, if at least one of the associated embeddings $e : \Gamma(e) \rightarrow \Gamma(t(e))$ and $\bar{e} : \Gamma(e) \rightarrow \Gamma(o(e))$ is an isomorphism. If S contains a trivial edge e_1 , to fix ideas, say $\Gamma(e_1)^{e_1} = \Gamma(t(e_1))$, then we contract the edge e_1 into the vertex $o(e_1)$ and re-define incidence and embeddings where necessary, to obtain a new graph of groups $(\Gamma'(-), X')$ with spanning tree S' in X' . More precisely, this means that we let

$$\begin{aligned} E(X') &= E(X) \setminus \{e_1, \bar{e}_1\}, \\ E(S') &= E(S) \setminus \{e_1, \bar{e}_1\}, \\ V(X') &= V(S') = V(X) \setminus \{t(e_1)\}, \end{aligned}$$

set

$$t'(e) := o(e_1), \quad \text{for } e \in E(X') \text{ with } t(e) = t(e_1),$$

and define new embeddings via

$$\Gamma(e) \xrightarrow{e} \Gamma(t(e_1)) \xrightarrow{e_1^{-1}} \Gamma(e_1) \xrightarrow{\bar{e}_1} \Gamma(o(e_1)) = \Gamma(t'(e)), \quad \text{for } e \in E(X') \text{ with } t(e) = t(e_1), \quad (5.2)$$

leaving incidence and embeddings unchanged wherever possible. Clearly, S' , the result of contracting the geometric edge $\{e_1, \bar{e}_1\}$ and deleting the vertex $t(e_1)$, is still a spanning

²The notation used in Equation (5.1) follows Serre; see Déf. 8 in [22, Sec. 4.4].

tree for X' and, if $(\Gamma(-), X)$ has property (F_1) or (F_2) , then so does $(\Gamma'(-), X')$ by construction.

It remains to see that the fundamental group of the new graph of groups $(\Gamma'(-), X')$ is isomorphic to Γ . The fundamental group

$$\pi_1(\Gamma(-), X, S)$$

of the graph of groups $(\Gamma(-), X)$ at the spanning tree S is generated by the groups $\Gamma(v)$ for $v \in V(X)$ plus extra generators γ_e for $e \in \mathcal{O}(X) - E(S)$, where $\mathcal{O}(X)$ is any orientation of X , subject to the relations

$$a^e = a^{\bar{e}}, \quad \text{for } e \in \mathcal{O}(S) \text{ and } a \in \Gamma(e), \quad (5.3)$$

$$\gamma_e a^e \gamma_e^{-1} = a^{\bar{e}}, \quad \text{for } e \in \mathcal{O}(X) - E(S) \text{ and } a \in \Gamma(e), \quad (5.4)$$

where $\mathcal{O}(S)$ is the orientation of the tree S induced by $\mathcal{O}(X)$, with a corresponding presentation for $\pi_1(\Gamma'(-), X', S')$; see §5.1 in [22, Chap. I]. The relations (5.3) corresponding to the geometric edge $\{e_1, \bar{e}_1\}$ identify $\Gamma(t(e_1))$ isomorphically with a subgroup of $\Gamma(o(e_1))$; we can thus delete the generators $\gamma \in \Gamma(t(e_1))$ against those relations by Tietze moves. This yields a presentation for $\pi_1(\Gamma(-), X, S)$ with the same set of generators as $\pi_1(\Gamma'(-), X', S')$. Moreover, those relations (5.3)–(5.4) coming from edges e with $t(e) = t(e_1)$ have to be re-expressed in terms of elements of $\Gamma(o(e_1))$, which leads exactly to the corresponding relations of $\pi_1(\Gamma'(-), X', S')$ obtained by extending the embedding $e : \Gamma(e) \rightarrow \Gamma(t(e_1))$ in the natural way as given in (5.2). Hence, $\pi_1(\Gamma(-), X, S) \cong \pi_1(\Gamma'(-), X', S')$. Since $V(X)$ is finite, the tree S is finite; thus, proceeding in the manner described, we obtain, after finitely many steps, a graph of groups $(\Delta(-), Y)$ with fundamental group Γ and a spanning tree T in Y without trivial edges, such that $(\Delta(-), Y)$ enjoys the finiteness properties $(F_1), (F_2)$ whenever $(\Gamma(-), X)$ does. \square

6. A GRAPH-THEORETIC LEMMA

The following auxiliary result, which is of an entirely graph-theoretic nature, will be used frequently in the next two sections.

Lemma 7. *Let T be a tree, and let $v_0 \in V(T)$ be any vertex. Then there exists one, and only one, orientation $\mathcal{O}(T)$ of T , such that the assignment $e \mapsto t(e)$ defines a bijection $\psi_{v_0} : \mathcal{O}(T) \rightarrow V(T) \setminus \{v_0\}$. This orientation is obtained by orienting each geometric edge so as to point away from the root v_0 ; that is, travelling along an oriented edge, the distance from v_0 in the path metric always increases.*

Lemma 7 is easy to show, even in this generality. Moreover, for our present purposes, the trees considered will all be finite, in which case the assertion of Lemma 7 may be proved by a straightforward induction on $|V(T)|$, which we sketch briefly: by our condition on the map ψ_{v_0} , all (geometric) edges incident with v_0 will have to be oriented away from the root v_0 . Delete v_0 together with edges incident to v_0 . The result is a disjoint union of finitely many subtrees, in which we choose the (previous) neighbours of v_0 as new roots. An application of the induction hypothesis to these rooted subtrees now finishes the proof.

In what follows, the orientation of a tree T with respect to a base point v_0 described in Lemma 7 will be denoted by $\mathcal{O}_{v_0}(T)$.

7. CLASSIFYING VIRTUALLY FREE GROUPS OF SMALL RANK

Let Γ be a finitely generated virtually free group, and let $\mu(\Gamma)$ be its free rank, as defined in Section 2. Then Γ is finite if $\mu(\Gamma) = 0$, virtually infinite-cyclic if $\mu(\Gamma) = 1$, and large in the sense of Pride's preorder on groups if $\mu(\Gamma) \geq 2$. Virtually infinite-cyclic groups play a certain role in topology as they are precisely the finitely generated groups with two ends. Their structure is well-known; cf. [24, 5.1] or [25, Lemma 4.1]. Here, we shall give a short proof of the corresponding result based on the tools developed in Sections 5 and 6.

Proposition 8. *A virtually infinite-cyclic group Γ falls into one of the following two classes:*

- (i) Γ has a finite normal subgroup with infinite-cyclic quotient.
- (ii) Γ is a free product $\Gamma = G_1 * G_2$ of two finite groups G_1 and G_2 , with an amalgamated subgroup A of index 2 in both factors.

Remark 9. In Part (ii) of Proposition 8, A is a finite normal subgroup of Γ with quotient the infinite dihedral group $C_2 * C_2$.

Proof of Proposition 8. Let $(\Gamma(-), X)$ be a finite graph of finite groups with fundamental group Γ and spanning tree T , chosen according to Lemma 6. The reader should observe that the assumption that Γ is virtually infinite-cyclic in combination with (2.7) implies that $\chi(\Gamma) = 0$.

If $|V(X)| = 1$, $V(X) = \{v\}$ say, then the above observation together with Formula (2.2) show that X has exactly one geometric edge $\{e, \bar{e}\}$, and that the associated embedding $e : \Gamma(e) \rightarrow \Gamma(v)$ is an isomorphism. Hence, $\Gamma(v) \trianglelefteq \Gamma$ and $\Gamma/\Gamma(v) \cong C_\infty$, which gives the desired result in case (i).

If $|V(X)| > 1$, we choose an edge $e_1 \in E(T)$, introduce the orientation $\mathcal{O}_{v_0}(T)$ with respect to the base point $v_0 = o(e_1)$, extend it to an orientation $\mathcal{O}(X)$ of X , and let $v_1 = t(e_1)$. We then split the Euler characteristic of Γ as follows:

$$0 = \chi(\Gamma) = \sum_{\substack{v \in V(X) \\ v \neq v_0, v_1}} \frac{1}{|\Gamma(v)|} - \sum_{\substack{e \in \mathcal{O}_{v_0}(T) \\ e \neq e_1}} \frac{1}{|\Gamma(e)|} + \left(\frac{1}{|\Gamma(v_0)|} + \frac{1}{|\Gamma(v_1)|} - \frac{1}{|\Gamma(e_1)|} \right) - \sum_{e \in \mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T)} \frac{1}{|\Gamma(e)|}. \quad (7.1)$$

By the normalisation condition (5.1) on $(\Gamma(-), X)$, we have

$$2|\Gamma(e_1)| \leq \gamma := \min \{|\Gamma(v_0)|, |\Gamma(v_1)|\},$$

so

$$\frac{1}{|\Gamma(v_0)|} + \frac{1}{|\Gamma(v_1)|} - \frac{1}{|\Gamma(e_1)|} \leq \frac{2}{\gamma} - \frac{1}{|\Gamma(e_1)|} \leq 0. \quad (7.2)$$

Clearly, equality in (7.2) occurs if, and only if, $\Gamma(e_1)$ is of index 2 in both $\Gamma(v_0)$ and $\Gamma(v_1)$. Similarly, by the normalisation condition (5.1) and Lemma 7,

$$\sum_{\substack{v \in V(X) \\ v \neq v_0, v_1}} \frac{1}{|\Gamma(v)|} - \sum_{\substack{e \in \mathcal{O}_{v_0}(T) \\ e \neq e_1}} \frac{1}{|\Gamma(e)|} = \sum_{\substack{e \in \mathcal{O}_{v_0}(T) \\ e \neq e_1}} \left(\frac{1}{|\Gamma(t(e))|} - \frac{1}{|\Gamma(e)|} \right) \leq 0,$$

with equality if, and only if, $\mathcal{O}_{v_0}(T) = \{e_1\}$. Also, trivially, the last sum on the right-hand side of (7.1) is non-negative, and vanishes if, and only if, $\mathcal{O}(X) = \mathcal{O}_{v_0}(T)$. Given this discussion, we conclude from (7.1) that $\Gamma = \Gamma(v_0) \underset{\Gamma(e_1)}{*} \Gamma(v_1)$, the amalgam being formed with respect to the embeddings $e_1 : \Gamma(e_1) \rightarrow \Gamma(v_1)$ and $\bar{e}_1 : \Gamma(e_1) \rightarrow \Gamma(v_0)$, and that $(\Gamma(v_0) : \Gamma(e_1)^{\bar{e}_1}) = 2 = (\Gamma(v_1) : \Gamma(e_1)^{e_1})$, whence the result in case (ii). \square

Corollary 10. *If Γ is virtually infinite-cyclic, then the function $f_\lambda(\Gamma)$ is constant. More precisely, we have $f_\lambda(\Gamma) = m_\Gamma$ for $\lambda \geq 1$ in Case (i) of Proposition 8, while in Case (ii) we have $f_\lambda(\Gamma) = |A| = m_\Gamma/2$.*

Proof. If Γ is as described in Case (i) of Proposition 8, then (2.5) shows that $g_\lambda(\Gamma) = 1$ for $\lambda \geq 0$, leading to $f_\lambda(\Gamma) = m_\Gamma$ for all $\lambda \geq 1$ by (2.4) and an immediate induction on λ .

For Γ as in Case (ii), Equation (2.5) yields

$$g_\lambda(\Gamma) = 2^{-2\lambda} \binom{2\lambda}{\lambda}, \quad \lambda \geq 0.$$

By the binomial theorem applied to the generating function $G_\Gamma(z)$ of the $g_\lambda(\Gamma)$'s, we obtain $G_\Gamma(z) = (1 - z)^{-1/2}$, which transforms into the relation

$$F_\Gamma(z) = \frac{|m_\Gamma|}{2(1 - z)} = \frac{|A|}{1 - z}$$

via (2.6). The desired result follows from this last equation by comparing coefficients. \square

By an argument similar to that in the proof of Proposition 8, again based on Lemmas 6 and 7, one also obtains a structural classification of the virtually free groups Γ of free rank $\mu(\Gamma) = 2$. We confine ourselves to stating the result, leaving details of the (straightforward if somewhat cumbersome) proof to the interested reader.

Proposition 11. *A virtually free group Γ of rank $\mu(\Gamma) = 2$ falls into one of the following five classes:*

- (i) Γ is an HNN-extension $\Gamma = G \underset{S, \sigma}{*}$ with finite base group G and $(G : S) = 2$.
- (ii) Γ contains a finite normal subgroup G with quotient $\Gamma/G \cong F_2$ free of rank 2.
- (iii) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ of two finite groups G_i with an amalgamated subgroup S , whose indices $(G_i : S)$ satisfy one of the conditions
 - (iii)₁ $\{(G_1 : S), (G_2 : S)\} = \{2, 3\}$,
 - (iii)₂ $(G_1 : S) = 3 = (G_2 : S)$,
 - (iii)₃ $\{(G_1 : S), (G_2 : S)\} = \{2, 4\}$.
- (iv) Γ is of the form $\Gamma = (G_1 \underset{G_{12}}{*} G_2) \underset{G_{23}}{*} G_3$ with finite factors G_i and indices $(G_1 : G_{12}) = 2 = (G_2 : G_{12})$ and $(G_2 : G_{23}) = 2 = (G_3 : G_{23})$.
- (v) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$, where G_1 contains a finite normal subgroup H_1 with $S \leq H_1$ and $G_1/H_1 \cong C_\infty$, G_2 is finite, and $(H_1 : S) = 2 = (G_2 : S)$.

8. SOME CRITERIA FOR A VIRTUALLY FREE GROUP TO BE ‘LARGE’

Our next result collects together a number of equivalent conditions on a finitely generated virtually free group Γ which all say, in one way or another, that Γ is ‘large’ in some particular sense. Perhaps the most obvious condition in this direction is given by Pride’s concept of being ‘as large as a free group of rank 2’. The concept of ‘largeness’ for groups, first introduced by S. Pride in [18], and further developed in [5], depends on a certain preorder \preceq on the class of groups, defined in [5] as follows: let G and H be groups. Then we write $H \preceq G$, if there exist

- (a) a subgroup G^0 of finite index in G ;
- (b) a subgroup H^0 of finite index in H , and a finite normal subgroup N^0 of H^0 ;
- (c) a homomorphism from G^0 onto H^0/N^0 .

We write $H \sim G$ if $H \preceq G$ and $G \preceq H$, and we denote by $[G]$ the equivalence class of the group G under \sim . By abuse of notation, we also denote by \preceq the preorder induced on the class of equivalence classes of groups. The finitely generated groups which are ‘largest’ in Pride’s sense are the ones having a subgroup of finite index which can be mapped homomorphically onto the free group F_2 of rank 2.

Another, more topological, way of saying that a finitely generated virtually free group is ‘large’, is that it has infinitely many ends. Here, the number $e(\Gamma)$ of ends of a group Γ is defined as

$$e(\Gamma) = \begin{cases} \dim H^0(\Gamma, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Z}_2)/\mathbb{Z}_2\Gamma), & \text{if } \Gamma \text{ is infinite,} \\ 0, & \text{if } \Gamma \text{ is finite.} \end{cases}$$

The reader is referred to [2] or [3, Sec. 2] for an introduction to the theory of ends of a group from an algebraic point of view; for a discussion from a more topological viewpoint, see, for instance, [7], [6], or [23].

Criterion (vii) below will play a role in the proof of Lemma 13, which is the main tool in establishing Theorem 2. In item (vi), the symbol $s_m(\Gamma)$ denotes the number of subgroups of index m in Γ .

Lemma 12. *Let Γ be a finitely generated virtually free group, and let $(\Gamma(-), X)$ be a finite graph of finite groups with fundamental group Γ , chosen so as to satisfy the normalisation condition (5.1) of Lemma 6. Then the following assertions on Γ are equivalent:*

- (i) $\chi(\Gamma) < 0$.
- (ii) $\mu(\Gamma) \geq 2$.
- (iii) Γ has infinitely many ends.
- (iv) The function $f_\lambda(\Gamma)$ is strictly increasing.
- (v) $\Gamma \sim F_2$ in the sense of Pride’s preorder \preceq on groups, where F_2 denotes the free group of rank 2.
- (vi) Γ has fast subgroup growth in the sense that the inequality $s_{nj}(\Gamma) \geq c \cdot n!$ holds for some fixed positive integer j , some constant $c > 0$, and all $n \geq 1$.

- (vii) If X has only one vertex v , then either X has more than one geometric edge, or $E(X) = \{e_1, \bar{e}_1\}$ and $(\Gamma(v) : \Gamma(e_1)^{e_1}) \geq 2$; if $|V(X)| \geq 2$, then X is not a tree, or X is a tree with more than one geometric edge, or $E(X) = \{e_1, \bar{e}_1\}$ and $\chi(\Gamma_0) < 0$, where $\Gamma_0 := \Gamma_{o(e_1)} *_{\Gamma(e_1)} \Gamma_{t(e_1)}$.

Proof. (i) \Leftrightarrow (ii). This is immediate from Formula (2.7) plus the fact that $\mu(\Gamma)$ is integral.

(ii) \Leftrightarrow (iii). This follows from [3, Prop. 2.1] (i.e., the fact that the number of ends is invariant when passing to a subgroup of finite index) and Examples 1 and 2 in [3] computing the number of ends of a free product, respectively of C_∞ .

(ii) \Leftrightarrow (iv). This follows from [14, Theorem 4] in conjunction with Corollary 10.

(ii) \Rightarrow (v). If $\mu(\Gamma) \geq 2$, then Γ contains a free group F of rank at least 2, with $(\Gamma : F) = m_\Gamma < \infty$; in particular, $F_2 \preceq \Gamma$. Since $[F_2]$ is largest with respect to the preorder \preceq among all equivalence classes of finitely generated groups, we also have $\Gamma \preceq F_2$, so $\Gamma \sim F_2$, as claimed.

(v) \Rightarrow (vi). Suppose that $\Gamma \sim F_2$. Then there exists a subgroup $\Delta \leq \Gamma$ of index $(\Gamma : \Delta) = j < \infty$ and a surjective homomorphism $\varphi : \Delta \rightarrow F_2$. From this plus Newman's asymptotic estimate [17, Theorem 2]

$$s_n(F_r) \sim n(n!)^{r-1} \text{ as } n \rightarrow \infty, \quad r \geq 2,$$

it follows that

$$s_{jn}(\Gamma) \geq s_n(\Delta) \geq s_n(F_2) \geq c \cdot n \cdot n! \geq c \cdot n!$$

for $n \geq 1$ and some constant $c > 0$, whence (vi).

(vi) \Rightarrow (ii). If $\mu(\Gamma) \leq 1$, then either Γ is finite, so $s_n(\Gamma) = 0$ for sufficiently large n , or Γ is virtually infinite-cyclic, implying

$$s_n(\Gamma) \leq n^\alpha, \quad n \geq 1,$$

for some constant α , by [12, Cor. 1.4.3]; see also [20]. In both cases, Condition (vi) does not hold.

(ii) \Leftrightarrow (vii). This follows by splitting the Euler characteristic $\chi(\Gamma)$ as in the proof of Proposition 8, making use of Lemmas 6 and 7. \square

9. THE MAIN LEMMA

For a positive integer n and a prime number p , denote by $v_p(n)$ the p -adic valuation of n , that is, the exponent of the highest p -power dividing n .

Lemma 13. *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) \geq 2$, and let p be a prime number not dividing m_Γ . Then we have*

$$v_p(g_\lambda(\Gamma)) \geq v_p(\lambda!), \quad \lambda \geq 0. \quad (9.1)$$

In particular, the function $v_p(g_\lambda(\Gamma))$ is non-negative, and unbounded as $\lambda \rightarrow \infty$.

Proof. Let $(\Gamma(-), X)$ be a Stallings decomposition of Γ , chosen according to Lemma 6, and let $\mathcal{O}(X)$ be any orientation of the graph X . Then, by Formula (2.5), plus the fact that $p \nmid m_\Gamma$, we have

$$v_p(g_\lambda(\Gamma)) = \sum_{e \in \mathcal{O}(X)} v_p((\lambda m_\Gamma / |\Gamma(e)|)!) - \sum_{v \in V(X)} v_p((\lambda m_\Gamma / |\Gamma(v)|)!), \quad \lambda \geq 0. \quad (9.2)$$

We now distinguish two cases depending on whether $|V(X)| = 1$ or $|V(X)| \geq 2$.

(a) $V(X) = \{v\}$. Let $\mathcal{O}(X) = \{e_1, \dots, e_r\}$. Then $m_\Gamma = |\Gamma(v)|$, and (9.2) becomes

$$v_p(g_\lambda(\Gamma)) = \sum_{\rho=1}^r v_p((\lambda|\Gamma(v)|/|\Gamma(e_\rho)|)!) - v_p(\lambda!), \quad \lambda \geq 0.$$

Since $\mu(\Gamma) \geq 2$ by hypothesis, the implication (ii) \Rightarrow (vii) of Lemma 12 tells us that either $r \geq 2$, or $r = 1$ and $2|\Gamma(e_1)| \leq |\Gamma(v)|$. Since $|\Gamma(e_\rho)| \leq |\Gamma(v)|$, we conclude that

$$v_p(g_\lambda(\Gamma)) \geq \begin{cases} v_p(\lambda!), & r \geq 2, \\ v_p\left(\binom{2\lambda}{\lambda}\right) + v_p(\lambda!), & r = 1, \end{cases} \geq v_p(\lambda!), \quad \lambda \geq 0.$$

(b) $|V(X)| \geq 2$. Let T be a spanning tree in X , let $e_1 \in E(T)$ be any edge, and let $\mathcal{O}(X)$ be an orientation of X extending $\mathcal{O}_{v_0}(T)$, where $v_0 = o(e_1)$. Let $v_1 = t(e_1)$ and let Γ_0 be as in Lemma 12(vii). Rewriting (9.2) by means of Legendre's formula for the p -part of factorials, and estimating resulting quantities by means of the inequality

$$\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor, \quad \text{for } x, y \in \mathbb{R},$$

we get

$$\begin{aligned} v_p(g_\lambda(\Gamma)) &= v_p((\lambda m_\Gamma / |\Gamma(e_1)|)!) - v_p((\lambda m_\Gamma / |\Gamma(v_0)|)!) - v_p((\lambda m_\Gamma / |\Gamma(v_1)|)!) \\ &\quad + \sum_{e \in \mathcal{O}_{v_0}(T) \setminus \{e_1\}} \left(v_p((\lambda m_\Gamma / |\Gamma(e)|)!) - v_p((\lambda m_\Gamma / |\Gamma(t(e))|)!) \right) \\ &\quad + \sum_{e \in \mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T)} v_p((\lambda m_\Gamma / |\Gamma(e)|)!) \\ &= \sum_{\mu \geq 1} \left(\left\lfloor \frac{\lambda m_\Gamma / |\Gamma(e_1)|}{p^\mu} \right\rfloor - \left\lfloor \frac{\lambda m_\Gamma / |\Gamma(v_0)|}{p^\mu} \right\rfloor - \left\lfloor \frac{\lambda m_\Gamma / |\Gamma(v_1)|}{p^\mu} \right\rfloor \right. \\ &\quad + \sum_{e \in \mathcal{O}_{v_0}(T) \setminus \{e_1\}} \left(\left\lfloor \frac{\lambda m_\Gamma / |\Gamma(e)|}{p^\mu} \right\rfloor - \left\lfloor \frac{\lambda m_\Gamma / |\Gamma(t(e))|}{p^\mu} \right\rfloor \right) \\ &\quad \left. + \sum_{e \in \mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T)} \left\lfloor \frac{\lambda m_\Gamma / |\Gamma(e)|}{p^\mu} \right\rfloor \right) \\ &\geq \sum_{\mu \geq 1} \left(\left\lfloor \frac{-\lambda m_\Gamma \chi(\Gamma_0)}{p^\mu} \right\rfloor + \sum_{e \in \mathcal{O}_{v_0}(T) \setminus \{e_1\}} \left\lfloor \frac{\lambda m_\Gamma}{p^\mu} \left(\frac{1}{|\Gamma(e)|} - \frac{1}{|\Gamma(t(e))|} \right) \right\rfloor \right. \\ &\quad \left. + \sum_{e \in \mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T)} \left\lfloor \frac{\lambda m_\Gamma / |\Gamma(e)|}{p^\mu} \right\rfloor \right). \end{aligned} \tag{9.3}$$

By the normalisation condition (5.1) on $(\Gamma(-), X)$, we have $\chi(\Gamma_0) \leq 0$ and

$$2|\Gamma(e)| \leq |\Gamma(t(e))|, \quad \text{for } e \in \mathcal{O}_{v_0}(T) \setminus \{e_1\},$$

so that the inequality for $v_p(g_\lambda(\Gamma))$ resulting from (9.3) entails

$$\begin{aligned}
v_p(g_\lambda(\Gamma)) &\geq v_p((\lambda m_\Gamma m_{\Gamma_0}^{-1}(\mu(\Gamma_0) - 1))!) + \sum_{e \in \mathcal{O}_{v_0}(T) \setminus \{e_1\}} v_p((\lambda m_\Gamma / |\Gamma(t(e))|)!) \\
&\quad + \sum_{e \in \mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T)} v_p((\lambda m_\Gamma / |\Gamma(e)|)!). \quad (9.4)
\end{aligned}$$

By hypothesis, we have $\mu(\Gamma) \geq 2$ hence, again by the implication (ii) \Rightarrow (vii) of Lemma 12, at least one of the assertions

- (1) $\mathcal{O}(X) \setminus \mathcal{O}_{v_0}(T) \neq \emptyset$,
- (2) $|\mathcal{O}_{v_0}(T)| \geq 2$,
- (3) $\mu(\Gamma_0) \geq 2$

holds true. It follows now from (9.4) that the estimate (9.1) also holds in Case (b), finishing the proof of the lemma. \square

Remark 14. It is easy to see that the estimate (9.1) in Lemma 13 is, in general, best possible. For instance, let $\Gamma = \text{PSL}_2(\mathbb{Z}) \cong C_2 * C_3$. Then

$$g_\lambda(\Gamma) = \frac{(6\lambda)!}{(2\lambda)!(3\lambda)!2^{3\lambda}3^{2\lambda}},$$

and, for a prime $p \geq 7$ and $\lambda = p^r$ with $r \geq 0$, we have

$$v_p(g_\lambda(\Gamma)) = \frac{p^r - 1}{p - 1} = v_p(\lambda!).$$

10. PROOF OF THEOREM 2

Theorem 2 follows easily from Lemmas 5 and 13. Indeed, if $\mu(\Gamma) = 0$, then Γ is finite, we have $m_\Gamma = |\Gamma|$, and thus $f_1(\Gamma) = 1$ and $f_\lambda(\Gamma) = 0$ for $\lambda \geq 2$, so that $f_\lambda(\Gamma)$ is ultimately periodic with period and preperiod equal to 1 modulo any prime power. If $\mu(\Gamma) = 1$, then, by Corollary 10, $f_\lambda(\Gamma)$ is constant, thus purely periodic with period equal to 1, again modulo any prime power. Now suppose that $\mu(\Gamma) \geq 2$. Given a positive integer α , let $\lambda_0(\alpha)$ be chosen according to Lemma 13 such that $v_p(g_\lambda(\Gamma)) \geq \alpha$ for all $\lambda \geq \lambda_0(\alpha)$ and $v_p(g_{\lambda_0(\alpha)-1}(\Gamma)) < \alpha$. Then consider Equation (2.4) for $\lambda \geq \lambda_0(\alpha)$. All summands on the left-hand side corresponding to indices $\mu \geq \lambda_0(\alpha)$ will vanish modulo p^α , as does the right-hand side, and we obtain the congruence

$$f_{\lambda+\lambda_0(\alpha)}(\Gamma) \equiv -(g_1(\Gamma)f_{\lambda+\lambda_0(\alpha)-1}(\Gamma) + \cdots + g_{\lambda_0(\alpha)-1}(\Gamma)f_{\lambda+1}(\Gamma)) \pmod{p^\alpha}, \quad \lambda \geq 0. \quad (10.1)$$

Applying Lemma 5 with $\Lambda = \mathbb{Z}/p^\alpha\mathbb{Z}$ and $\mathcal{S} = (f_{\lambda+1}(\Gamma))_{\lambda \geq 0}$, we find that, in this last case, $f_\lambda(\Gamma)$ is ultimately periodic modulo p^α with least period $\omega_0 < p^{\alpha(\lambda_0(\alpha)-1)}$, whence the result.

11. PROOF OF THEOREM 3

If $p \mid m_\Gamma$, then, by [16, Eq. (3)], the generating function $F_\Gamma(z)$ satisfies the congruence

$$F_\Gamma(z) = z^{\mu_p(\Gamma)} F_\Gamma^{\mu_p(\Gamma)}(z) (z^{p-1} F_\Gamma(z)^{p-1} - 1)^{(\mu(\Gamma) - \mu_p(\Gamma))/(p-1)} \pmod{p}. \quad (11.1)$$

As is argued in [16], if $\mu_p(\Gamma) > 0$, then it is obvious from this congruence that $F_\Gamma(z) = 0$ modulo p .

We shall now demonstrate by an induction on α that, for all integers $\alpha \geq 1$, the generating function $F_\Gamma(z)$ is rational when coefficients are reduced modulo p^α . For $\alpha = 1$ this is true due to the above remark.

Let us now suppose that we have already shown that $F_\Gamma(z)$ is rational when coefficients are reduced modulo p^α , say $F_\Gamma(z) = R(z)$ modulo p^α , for some rational function $R(z)$ over the integers whose denominator is not divisible by p . By [16, Eq. (12)] and (11.1), we know that

$$F_\Gamma(z) = z^{\mu_p(\Gamma)} F_\Gamma^{\mu_p(\Gamma)}(z) (z^{p-1} F_\Gamma(z)^{p-1} - 1)^{(\mu(\Gamma) - \mu_p(\Gamma))/(p-1)} + p \cdot \mathcal{P}(z, F_\Gamma(z), F'_\Gamma(z), F''_\Gamma(z), \dots, F_\Gamma^{(\mu(\Gamma)-1)}(z)), \quad (11.2)$$

where $\mathcal{P}(z, F_\Gamma(z), F'_\Gamma(z), \dots, F_\Gamma^{(\mu(\Gamma)-1)}(z))$ is a polynomial in $z, F_\Gamma(z), F'_\Gamma(z), \dots, F_\Gamma^{(\mu(\Gamma)-1)}(z)$ over the *rational*s. However, it is proven in [16, Sections 3 and 5] that, if $p \mid m_\Gamma$, the rational coefficients can be written with denominators which are relatively prime to p , a fact that we shall tacitly use in the sequel.

We now make the Ansatz $F_\Gamma(z) = R(z) + p^\alpha Y(z)$, for some unknown formal power series $Y(z)$, we substitute in (11.2), and then consider the result modulo $p^{\alpha+1}$. Since

$$(R(z) + p^\alpha Y(z))^e = R^e(z) + ep^\alpha R^{e-1}(z)Y(z) \quad \text{modulo } p^{\alpha+1},$$

we arrive at the congruence

$$\begin{aligned} R(z) + p^\alpha Y(z) &= \sum_{i=0}^M (-1)^{M-i} \binom{M}{i} z^{\mu_p(\Gamma) + i(p-1)} \\ &\quad \cdot (R^{\mu_p(\Gamma) + i(p-1)}(z) + p^\alpha (\mu_p(\Gamma) + i(p-1)) R^{\mu_p(\Gamma) + i(p-1)-1}(z) Y(z)) \\ &\quad + p \cdot \mathcal{P}(z, R(z), R'(z), \dots, R^{(\mu(\Gamma)-1)}(z)) \quad \text{modulo } p^{\alpha+1}, \end{aligned} \quad (11.3)$$

where M is short for $(\mu(\Gamma) - \mu_p(\Gamma))/(p-1)$. By rearranging terms, this turns into

$$\begin{aligned} p^\alpha Y(z) &\cdot \left(-1 + \sum_{i=0}^M (-1)^{M-i} \binom{M}{i} z^{\mu_p(\Gamma) + i(p-1)} (\mu_p(\Gamma) + i(p-1)) R^{\mu_p(\Gamma) + i(p-1)-1}(z) \right) \\ &= R(z) - \sum_{i=0}^M (-1)^{M-i} \binom{M}{i} z^{\mu_p(\Gamma) + i(p-1)} R^{\mu_p(\Gamma) + i(p-1)}(z) \\ &\quad - p \cdot \mathcal{P}(z, R(z), R'(z), \dots, R^{(\mu(\Gamma)-1)}(z)) \quad \text{modulo } p^{\alpha+1}. \end{aligned} \quad (11.4)$$

By induction hypothesis, the right-hand side is divisible by p^α . We may hence divide both sides by p^α , to obtain the congruence

$$Y(z) \cdot \left(-1 + \sum_{i=0}^M (-1)^{M-i} \binom{M}{i} z^{\mu_p(\Gamma) + i(p-1)} (\mu_p(\Gamma) - i) R^{\mu_p(\Gamma) + i(p-1)-1}(z) \right) = S(z) \quad \text{modulo } p,$$

where $S(z)$ can be written as an explicit rational function in z over the integers with denominator not divisible by p . If we remember that, from the base case of the induction

(see the sentence below (11.1)), it follows that $R(z) = 0$ modulo p , then we see that the above congruence simplifies further to

$$Y(z) \cdot \left(-1 + (-1)^M \mu_p(\Gamma) z^{\mu_p(\Gamma)} R^{\mu_p(\Gamma)-1}(z) \right) = S(z) \quad \text{modulo } p. \quad (11.5)$$

We can therefore determine $Y(z)$ modulo p by dividing both sides of the congruence by the term in parentheses on the left-hand side. Hence $Y(z)$ is a rational function modulo p , and therefore $F_\Gamma(z) = R(z) + p^\alpha Y(z)$ is rational modulo $p^{\alpha+1}$. This concludes the induction argument.

However, the additional assertions in Theorem 3 are now also obvious: if $\mu_p(\Gamma) = 1$, then every time we divide by $1 - (-1)^M z$, and the induction hypothesis guarantees that all denominators of fractions in the congruence (11.5) are a power of $1 - (-1)^M z$, cf. the implicit definition of $S(z)$ via (11.4). On the other hand, if $\mu_p(\Gamma) \geq 2$, then as induction hypothesis let us suppose that $R(z)$ is actually a *polynomial* modulo p^α . Again using the fact that $R(z) = 0$ modulo p , we see that, since $\mu_p(\Gamma) + i(p-1) > 0$ for all $i \geq 0$ in the current case, the congruence (11.5) reduces to

$$-Y(z) = S(z) \quad \text{modulo } p.$$

Here, the rational function $S(z)$ is actually a *polynomial*. Consequently, $Y(z)$ is a polynomial modulo p , and thus $F_\Gamma(z) = R(z) + p^\alpha Y(z)$ is a polynomial modulo $p^{\alpha+1}$. This completes the proof of the theorem.

12. PROOF OF THEOREM 1

(i) *implies* (ii). This is obvious.

(ii) *implies* (iii). We may have $p \nmid m_\Gamma$ (Case (iii)₁), or $p \mid m_\Gamma$ and $\mu_p(\Gamma) > 0$ (Case (iii)₂), or $p \mid m_\Gamma$ and $\mu_p(\Gamma) = 0$. Due to the definition (2.9) of μ_p , the condition $\mu_p(\Gamma) = 0$ already implies that $p \mid m_\Gamma$ (in the sum on the right-hand side of (2.9) only the term for $\kappa = m_\Gamma$ can be negative). Thus, it remains to consider the case where $\mu_p(\Gamma) = 0$. In this case, Theorem A(iii) in [16] says that either $\mu(\Gamma) = 0$, or $\mu(\Gamma) = 1$ and $p = 2$. In the former case, the group Γ is finite (Case (iii)₃), while in the latter case Γ is virtually infinite-cyclic (Case (iii)₄).

(iii) *implies* (i). We have to distinguish between the various subcases given in this item. In each case, we have to show that the generating function $F_\Gamma(z)$ is rational modulo p^α for every $\alpha \geq 1$.

Case (iii)₁. This is taken care of by Theorem 2.

Case (iii)₂. This is dealt with by Theorem 3.

Case (iii)₃. This is obvious since in this case $F_\Gamma(z) = 1$.

Case (iii)₄. Corollary 10 says that in this case the sequence of free subgroup numbers $f_\lambda(\Gamma)$ is constant. Consequently, the corresponding generating function $F_\Gamma(z)$ is rational even over the integers.

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